PRINCIPLES OF FINANCIAL ECONOMICS

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1

Equilibrium in Security Markets

1.1 Introduction

The analytical framework in the classical finance models discussed in this book is largely the same as in general equilibrium theory: agents, acting as price-takers, exchange claims on consumption to maximize their respective utilities. Because the focus in financial economics is somewhat different from that in mainstream economics, we will ask for greater generality in some directions while sacrificing generality in favor of simplification in other directions.

As an example of greater generality, it will be assumed that markets are incomplete: the Arrow–Debreu assumption of complete markets is an important special case, but in general it will not be assumed that agents can purchase any imaginable payoff pattern on securities markets. Another example is that uncertainty will always be explicitly incorporated in the analysis. It is not asserted that there is any special merit in doing so; the point is simply that the area of economics that deals with the same concerns as finance but concentrates on production rather than uncertainty has a different name (capital theory).

As an example of simplification, it will generally be assumed in this book that only one good is consumed and that there is no production. Again, the specialization to a single-good exchange economy is adopted only to focus attention on the concerns that are distinctive to finance rather than microeconomics, in which it is assumed that there are many goods (some produced), or capital theory, in which production economies are analyzed in an intertemporal setting.

In addition to those simplifications motivated by the distinctive concerns of finance, classical finance shares many of the same restrictions as Walrasian equilibrium analysis: agents treat the market structure as given, implying that no one tries to create new trading opportunities, and the abstract Walrasian auctioneer must be introduced to establish prices. Markets are competitive and free of transaction costs (except possibly costs of certain trading restrictions, as analyzed in Chapter 4), and they clear instantaneously. Finally, it is assumed that all agents have the same information. This last assumption largely defines the term “classical”; much of the best work now being done in finance assumes asymmetric information and therefore lies outside the framework of this book.

However, even students whose primary interest is in the economics of asymmetric information are well advised to devote some effort to understanding how financial markets work under symmetric information before passing to the much more difficult general case.

1.2 Security Markets

Securities are traded at date 0, and their payoffs are realized at date 1. Date 0, the present, is certain, whereas any of S states can occur at date 1, representing the uncertain future.

Security j is identified by its payoff x_j, an element of $\mathcal{R}^S$, where x_{js} denotes the payoff the holder of one share of security j receives in state s at date 1. Payoffs are in terms of the consumption good. They may be positive, zero, or negative. There exists a finite number J of securities with payoffs $x_1, \ldots, x_J,\ x_j \in \mathcal{R}^S$, taken as given.

The $J \times S$ matrix X of payoffs of all securities

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_J \end{bmatrix} \tag{1.1}$$

is the *payoff matrix*. Here, vectors x_j are understood to be row vectors. In general, vectors are understood to be either row vectors or column vectors, as the context requires.

A *portfolio* is composed of holdings of the J securities. These holdings may be positive, zero, or negative. A positive holding of a security means a long position in that security, whereas a negative holding means a short position (short sale). Thus, short sales are allowed (except in Chapters 4 and 7).

A portfolio is denoted by a J-dimensional vector h, where h_j denotes the holding of security j. The *portfolio payoff* is $\sum_j h_j x_j$ and can be represented as hX.

The set of payoffs available via trades in security markets is the *asset span* and is denoted by $\mathcal{M}$:

$$\mathcal{M} = \{z \in \mathcal{R}^S : z = hX \text{ for some } h \in \mathcal{R}^J\}. \tag{1.2}$$

Thus $\mathcal{M}$ is the subspace of $\mathcal{R}^S$ spanned by the security payoffs, that is, the row span of the payoff matrix X. If $\mathcal{M} = \mathcal{R}^S$, then markets are *complete*. If $\mathcal{M}$ is a proper subspace of $\mathcal{R}^S$, then markets are *incomplete*. When markets are complete, any date-1 consumption plan (that is, any element of $\mathcal{R}^S$) can be obtained as a portfolio payoff but perhaps not uniquely.

Theorem 1.2.1 *Markets are complete iff the payoff matrix X has rank S.*[1]

Proof: Asset span $\mathcal{M}$ equals the whole space $\mathcal{R}^S$ iff the equation $z = hX$, with J unknowns h_j, has a solution for every $z \in \mathcal{R}^S$. A necessary and sufficient condition for this is that X has rank S. □

A security is *redundant* if its payoff can be generated as the payoff of a portfolio of other securities. There are no redundant securities iff the payoff matrix X has rank J.

The prices of securities at date 0 are denoted by a J-dimensional vector $p = (p_1, \ldots, p_J)$. The price of portfolio h at security prices p is $ph = \sum_j p_j h_j$.

The *return* r_j on security j is its payoff x_j divided by its price p_j (assumed to be nonzero; the return on a payoff with zero price is undefined):

$$r_j = \frac{x_j}{p_j}. \tag{1.3}$$

Thus, "return" means gross return ("net return" equals gross return minus one). Throughout we will be working with gross returns.

Frequently the practice in the finance literature is to specify the asset span using the returns on the securities rather than their payoffs. The asset span is the subspace of $\mathcal{R}^S$ spanned by the returns on the securities.

The following example illustrates the concepts introduced above.

Example 1.2.2 Let there be three states and two securities. Security 1 is risk free and has payoff $x_1 = (1, 1, 1)$. Security 2 is risky with $x_2 = (1, 2, 2)$. The payoff matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

The asset span is $\mathcal{M} = \{(z_1, z_2, z_3) : z_1 = h_1 + h_2,\ z_2 = h_1 + 2h_2,\ z_3 = h_1 + 2h_2,\ \text{for some } (h_1, h_2)\}$ – a two-dimensional subspace of R^3. By inspection, $\mathcal{M} = \{(z_1, z_2, z_3) : z_2 = z_3\}$. At prices $p_1 = 0.8$ and $p_2 = 1.25$, security returns are $r_1 = (1.25, 1.25, 1.25)$ and $r_2 = (0.8, 1.6, 1.6)$. □

---

[1] Here and throughout this book, "A iff B," an abbreviation for "A if and only if B," has the same meaning as "A is equivalent to B" and as "for A to be true, B is a necessary and sufficient condition." Therefore, proving necessity in "A iff B" means proving "A implies B," whereas proving sufficiency means proving "B implies A."

1.3 Agents

In the most general case (pending discussion of the multidate model), agents consume at both dates 0 and 1. Consumption at date 0 is represented by the scalar c_0, whereas consumption at date 1 is represented by the S-dimensional vector $c_1 = (c_{11}, \ldots, c_{1S})$, where c_{1s} represents consumption conditional on state s. Consumption c_{1s} will be denoted by c_s when no confusion can result.

At times we will restrict the set of admissible consumption plans. The most common restriction will be that c_0 and c_1 be positive.[2] However, when using particular utility functions it is generally necessary to impose restrictions other than, or in addition to, positivity. For example, the logarithmic utility function presumes that consumption is strictly positive, whereas the quadratic utility function $u(c) = -\sum_{s=1}^{S}(c_s - \alpha)^2$ has acceptable properties only when $c_s \leq \alpha$. However, under the quadratic utility function, unlike the logarithmic function, zero or negative consumption poses no difficulties.

There is a finite number I of agents. Agent i's preferences are indicated by a continuous utility function $u^i : \mathcal{R}_+^{S+1} \to \mathcal{R}$, in the case in which admissible consumptions are restricted to be positive and $u^i(c_0, c_1)$ is the utility of consumption plan (c_0, c_1). Agent i's endowment is w_0^i at date 0 and w_1^i at date 1.

A *securities market economy* is an economy in which all agents' endowments lie in the asset span. In that case one can think of agents as endowed with initial portfolios of securities (see Section 1.7).

Utility function u is *increasing at date* 0 if $u(c_0', c_1) \geq u(c_0, c_1)$ whenever $c_0' \geq c_0$ for every c_1 and *increasing at date* 1 if $u(c_0, c_1') \geq u(c_0, c_1)$ whenever $c_1' \geq c_1$ for every c_0. It is *strictly increasing at date* 0 if $u(c_0', c_1) > u(c_0, c_1)$ whenever $c_0' > c_0$ for every c_1 and *strictly increasing at date* 1 if $u(c_0, c_1') > u(c_0, c_1)$ whenever $c_1' > c_1$ for every c_0. If u is (strictly) increasing at date 0 and at date 1, then u is (strictly) increasing.

Utility functions and endowments typically differ across agents; nevertheless, the superscript i will frequently be deleted when no confusion can result.

1.4 Consumption and Portfolio Choice

At date 0 agents consume their date-0 endowments less the value of their security purchases. At date 1 they consume their date-1 endowments plus their security payoffs. The agent's consumption-portfolio choice problem is

---

[2] Our convention on inequalities is as follows: for two vectors $x, y \in \mathcal{R}^n$,

$x \geq y$ means that $x_i \geq y_i \ \forall i$; x is greater than y,
$x > y$ means that $x \geq y$ and $x \neq y$; x is greater than but not equal to y,
$x \gg y$ means that $x_i > y_i \ \forall i$; x is strictly greater than y.

For a vector x, *positive* means $x \geq 0$, *positive and nonzero* means $x > 0$, and *strictly positive* means $x \gg 0$. These definitions apply to scalars as well. For scalars, "positive and nonzero" is equivalent to "strictly positive."

$$\max_{c_0, c_1, h} u(c_0, c_1) \tag{1.4}$$

subject to

$$c_0 \leq w_0 - ph \tag{1.5}$$

$$c_1 \leq w_1 + hX \tag{1.6}$$

and a restriction that consumption be positive, $c_0 \geq 0,\ c_1 \geq 0$, if that restriction is imposed.

When, as in Chapters 11 and 13, we want to analyze an agent's optimal portfolio abstracting from the effects of intertemporal consumption choice, we will consider a simplified model in which date-0 consumption does not enter the utility function. The agent's choice problem is then

$$\max_{c_1, h} u(c_1) \tag{1.7}$$

subject to

$$ph \leq w_0 \tag{1.8}$$

and

$$c_1 \leq w_1 + hX. \tag{1.9}$$

1.5 First-Order Conditions

If utility function u is differentiable, the first-order conditions for a solution to the consumption-portfolio choice problem (1.4)–(1.6) (if the constraint $c_0 \geq 0, c_1 \geq 0$ is imposed) are

$$\partial_0 u(c_0, c_1) - \lambda \leq 0, \qquad [\partial_0 u(c_0, c_1) - \lambda]c_0 = 0 \tag{1.10}$$

$$\partial_s u(c_0, c_1) - \mu_s \leq 0, \qquad [\partial_s u(c_0, c_1) - \mu_s]c_s = 0\ , \qquad \forall s \tag{1.11}$$

$$\lambda p = X\mu, \tag{1.12}$$

where λ and $\mu = (\mu_1, \ldots, \mu_S)$ are positive Lagrange multipliers.[3]

---

[3] If f is a function of a single variable, its first derivative is indicated $f'(x)$ or, when no confusion can result, f'. Similarly, the second derivative is indicated $f''(x)$ or f''. The partial derivative of a function f of two variables x and y with respect to the first variable is indicated $\partial_x f(x, y)$ or $\partial_x f$.

Frequently the function in question is a utility function u, and the argument is (c_0, c_1), where, as noted above, c_0 is a scalar and c_1 is an S-vector. In that case the partial derivative of the function u with respect to c_0 is denoted $\partial_0 u(c_0, c_1)$ or $\partial_0 u$, and the partial derivative with respect to c_s is denoted $\partial_s u(c_0, c_1)$ or $\partial_s u$. The vector of S partial derivatives with respect to c_s, for all s is denoted $\partial_1 u(c_0, c_1)$ or $\partial_1 u$.

Note that there exists the possibility of confusion: the subscript "1" can indicate either the vector of date-1 partial derivatives or the (scalar) partial derivative with respect to consumption in state 1. The context will always make the intended meaning clear.

If u is quasi-concave, then these conditions are sufficient as well as necessary. If it is assumed that the solution is interior and that $\partial_0 u > 0$, inequalities (1.10) and (1.11) are satisfied with equality. Then Eq. (1.12) becomes

$$p = X\frac{\partial_1 u}{\partial_0 u} \tag{1.13}$$

with typical equation

$$p_j = \sum_s x_{js}\frac{\partial_s u}{\partial_0 u}, \tag{1.14}$$

where we now – and henceforth – delete the argument of u in the first-order conditions. Equation (1.14) says that the price of security j (which is the cost in units of date-0 consumption of a unit increase in the holding of the jth security) is equal to the sum over states of its payoff in each state multiplied by the marginal rate of substitution between consumption in that state and consumption at date 0.

The first-order conditions for problem (1.7) with no consumption at date 0 are as follows:

$$\partial_s u - \mu_s \leq 0, \qquad (\partial_s u - \mu_s)c_s = 0, \quad \forall s \tag{1.15}$$

$$\lambda p = X\mu. \tag{1.16}$$

At an interior solution, Eq. (1.16) becomes

$$\lambda p = X\partial_1 u \tag{1.17}$$

with typical element

$$\lambda p_j = \sum_s x_{js}\partial_s u. \tag{1.18}$$

Because security prices are denominated in units of an abstract numeraire, all we can say about security prices is that they are proportional to the sum of marginal-utility-weighted payoffs.

1.6 Left and Right Inverses of a Matrix

The payoff matrix X has an inverse iff it is a square matrix ($J = S$) and of full rank. Neither of these properties is assumed to be true in general. However, even if X is not square, it may have a *left inverse,* defined as a matrix L that satisfies $LX = I_S$, where I_S is the $S \times S$ identity matrix. The left inverse exists iff X is of rank S, which occurs if $J \geq S$ and the columns of X are linearly independent. Iff the left inverse of X exists, the asset span $\mathcal{M}$ coincides with the date-1 consumption space $\mathcal{R}^S$, and thus markets are complete.

If markets are complete, the vectors of marginal rates of substitution of all agents (whose optimal consumption is interior) are the same and can be inferred uniquely from security prices. To see this, premultiply Eq. (1.13) by the left inverse L to obtain

$$Lp = \frac{\partial_1 u}{\partial_0 u}. \tag{1.19}$$

If markets are incomplete, the vectors of marginal rates of substitution may differ across agents.

Similarly, X may have a *right inverse*, which is defined as a matrix R that satisfies $XR = I_J$. The right inverse exists if X is of rank J, which occurs if $J \leq S$ and the rows of X are linearly independent. Then, no security is redundant. Any date-1 consumption plan c_1 such that $c_1 - w_1$ belongs to the asset span is associated with a unique portfolio

$$h = (c_1 - w_1)R, \tag{1.20}$$

which is derived by postmultiplying Eq. (1.6) by R.

The left and right inverses, if they exist, are given by

$$L = (X'X)^{-1}X' \tag{1.21}$$

$$R = X'(XX')^{-1}, \tag{1.22}$$

where the prime indicates transposition. As these expressions make clear, L exists iff $X'X$ is invertible, whereas R exists iff XX' is invertible.

The payoff matrix X is invertible iff both the left and right inverses exist. Under the assumptions thus far, none of the following four possibilities is ruled out:

1. Both left and right inverses exist.
2. The left inverse exists, but the right inverse does not exist.
3. The right inverse exists, but the left inverse does not exist.
4. Neither directional inverse exists.

1.7 General Equilibrium

An *equilibrium* in security markets consists of a vector of security prices p, a portfolio allocation $\{h^i\}$, and a consumption allocation $\{(c_0^i, c_1^i)\}$ such that (1) portfolio h^i and consumption plan (c_0^i, c_1^i) are a solution to agent i's choice problem (1.4) at prices p, and (2) markets clear; that is

$$\sum_i h^i = 0, \tag{1.23}$$

and

$$\sum_i c_0^i \leq \bar{w}_0 \equiv \sum_i w_0^i, \qquad \sum_i c_1^i \leq \bar{w}_1 \equiv \sum_i w_1^i. \tag{1.24}$$

The portfolio market-clearing condition (1.23) implies, by summing agents' budget constraints, the consumption market-clearing condition (1.24). If agents' utility functions are strictly increasing so that all budget constraints hold with equality, and if there are no redundant securities (X has a right inverse), then the converse is also true. If, on the other hand, there are redundant securities, then there are many portfolio allocations associated with a market-clearing consumption allocation. At least one of these portfolio allocations is market clearing.

In the simplified model in which date-0 consumption does not enter utility functions, each agent's equilibrium portfolio and date-1 consumption plan are a solution to the choice problem (1.7). Agents' endowments at date 0 are equal to zero, and thus there is zero demand and zero supply of date-0 consumption.

As the portfolio market-clearing condition (1.23) indicates, securities are in zero supply. This is consistent with the assumption that agents' endowments are in the form of consumption endowments. However, our modeling format allows consideration of the case in which agents have initial portfolios of securities and there is positive supply of securities. In that case, equilibrium portfolio allocation $\{h^i\}$ should be interpreted as an allocation of net trades in securities markets. To be more specific, suppose (in a securities market economy) that each agent's endowment at date 1 equals the payoff of an *initial portfolio* $\hat{h}^i$ so that $w_1^i = \hat{h}^i X$. Using total portfolio holdings, one can write an equilibrium as a vector of security prices p, an allocation of total portfolios $\{\bar{h}^i\}$, and a consumption allocation $\{(c_0^i, c_1^i)\}$ such that the net portfolio holding $h^i = \bar{h}^i - \hat{h}^i$ and consumption plan (c_0^i, c_1^i) are a solution to the problem (1.4) for each agent i, and

$$\sum_i \bar{h}^i = \sum_i \hat{h}^i, \tag{1.25}$$

and

$$\sum_i c_0^i \leq \sum_i w_0^i, \qquad \sum_i c_1^i \leq \sum_i \hat{h}^i X. \tag{1.26}$$

1.8 Existence and Uniqueness of Equilibrium

The existence of a general equilibrium in security markets is guaranteed under the standard assumptions of positivity of consumption and quasi-concavity of utility functions.

Theorem 1.8.1 *If each agent's admissible consumption plans are restricted to be positive, his utility function is strictly increasing and quasi-concave, his initial endowment is strictly positive, and a portfolio with positive and nonzero payoff exist, then an equilibrium in security markets exists.*

Although the proof is not given here, it can be found in the sources cited in the notes at the end of this chapter.

Without further restrictions on agents' utility functions, initial endowments or security payoffs, there may be multiple equilibrium prices and allocations in security markets. If all agents' utility functions are such that they imply gross substitutability between consumption at different states and dates, and if security markets are complete, then the equilibrium consumption allocation and prices are unique. This is because, as shown in Chapter 15, equilibrium allocations in complete security markets are the same as Walrasian equilibrium allocations. The corresponding equilibrium portfolio allocation is unique as long as there are no redundant securities. Otherwise, if there are redundant securities, then there are infinitely many portfolio allocations that generate the equilibrium consumption allocation.

1.9 Representative Agent Models

Many of the points to be made in this book are most simply illustrated using *representative agent models*: models in which all agents have identical utility functions and endowments. With all agents alike, security prices at which no agent wants to trade are equilibrium prices, for then markets clear. Equilibrium consumption plans equal endowments.

In representative agent models, specification of securities is unimportant, for in equilibrium agents are willing to consume their endowments regardless of which markets exist. It is often most convenient to assume complete markets so as to allow discussion of equilibrium prices of all possible securities.

1.10 Notes

As observed in the introduction, it is a good idea for the reader to make up and analyze as many examples as possible in studying financial economics. The question of how to represent preferences arises. It happens that a few utility functions are used in the large majority of cases in view of their convenient properties. Presentation of these utility functions is deferred to Chapter 9 because a fair amount of preliminary work is needed before these properties can be presented in a way that makes sense. However, it is worthwhile to find out what these utility functions are.

The purpose of specifying security payoffs is to determine the asset span $\mathcal{M}$. It was observed that the asset span can be specified using the returns on the securities rather than their payoffs. This requires the assumption that $\mathcal{M}$ does not consist of payoffs with zero price alone, for in that case returns are undefined. As long as $\mathcal{M}$ has a set of basis vectors of which at least one has nonzero price, then another basis of $\mathcal{M}$ can always be found of which all the vectors have nonzero price. Therefore, these can be rescaled to have unit price. It is important to bear in mind that returns are not simply an arbitrary rescaling of payoffs. Payoffs are given exogenously; returns, being payoffs divided by equilibrium prices, are endogenous.

The model presented in this chapter is based on the theory of general equilibrium as formulated by Arrow [1] and Debreu [3]. In some respects, the present treatment is more general than that of Arrow–Debreu; most significantly, we assume that agents trade securities in markets that may be incomplete, whereas Arrow and Debreu assumed complete markets. On the other hand, our specification involves a single good, whereas the Arrow–Debreu model allows for multiple goods. Accordingly, our framework can be seen as the general equilibrium model with incomplete markets (GEI) simplified to the case of a single good. See Geanakoplos [4] for a survey of the literature on GEI models; see also Magill and Quinzii [8] and Magill and Shafer [9].

The proof of Theorem 1.8.1 can be found in Milne [11], see also Geanakoplos and Polemarchakis [5]. Our maintained assumptions of symmetric information (agents anticipate the same state-contingent security payoffs) and a single good are essential for the existence of an equilibrium when short sales are allowed. An extensive literature is available on the existence of a security markets equilibrium when agents have different expectations about security payoffs. See Hart [7], Hammond [6], Nielsen [13], Page [14], and Werner [15]. On the other hand, the assumption of strictly positive endowments can be significantly weakened. Consumption sets other than the set of positive consumption plans can also be included (see Nielsen [13], Page [14], and Werner [15]). For discussions of the existence of an equilibrium in a model with multiple goods (GEI), see Geanakoplos [4] and Magill and Shafer [9].

A sufficient condition for satisfaction of the gross substitutes condition mentioned in Section 1.8 is that agents have strictly concave expects utility functions with common probabilities and with Arrow–Pratt measures of relative risk aversion (see Chapter 4) that are everywhere less than one. A few further results on uniqueness exist. It follows from results of Mitiushin and Polterovich [12] (in Russian) that if agents have strictly concave expected utility functions with common probabilities and relative risk aversion that is everywhere less than four, if their endowments are collinear (that is, each agent's endowment is a fixed proportion (the same in all states) of the aggregate endowment) and security markets are complete, then equilibrium is unique. See Mas-Colell [10] for a discussion of the Mitiushin-Polterovich result and of uniqueness generally. See also Dana [2] on uniqueness in financial models.

As observed in the introduction, throughout this book only exchange economies are considered. The reason is that production theory – or, in intertemporal economies, capital theory – does not lie within the scope of finance as usually defined, and not much is gained by combining exposition of the theory of asset pricing with that of resource allocation. The theory of the equilibrium allocation of resources is modeled by including production functions (or production sets) and assuming that agents have endowments of productive resources instead of, or in addition to, endowments of consumption goods. Because these production functions share most of the properties of utility functions, the theory of allocation of productive resources is similar to that of consumption goods.

In the finance literature there has been much discussion of the problem of determining firm behavior under incomplete markets when firms are owned by stockholders with different utility functions. There is, of course, no difficulty when markets are complete: even if stockholders have different preferences, they will agree that firms should maximize profit. However, when markets are incomplete and firm output is not in the asset span, firm output cannot be valued unambiguously. If this output is distributed to stockholders in proportion to their ownership shares, the stockholders will generally disagree about the ordering of different possible outputs.

This is not a genuine problem – at least in the kinds of economies modeled in this book. The reason is that in the framework considered here – in which all problems of scale economies, externalities, coordination, agency issues, incentives, and the like are ruled out – there is no reason for nontrivial firms to exist in the first place. As is well known, in such neoclassical production economies the zero-profit condition guarantees that there is no difference between an agent's renting out his or her own resource endowment and employing other agents' resources if it is assumed that all agents have access to the same technology. Therefore, there is no reason not to consider each owner of productive resources as operating his or her own firm. Of course, this is saying nothing more than that if firms play only a trivial role in the economy, then there can exist no nontrivial problem about what the firm should do. In a setting in which firms do play a nontrivial role, these issues of corporate governance become significant.

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